

A New Nonextensive Entropy

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We propose a new way of defining entropy of a system, which gives a general form which may be nonextensive as Tsallis entropy, but is linearly dependent on component entropies, like Renyi entropy, which is extensive. This entropy has a conceptually novel but simple origin and is mathematically easy to define by a very simple expression, though the probability distribution resulting from optimizing it gives rather complex, which is compared numerically with the other entropies. It may, therefore, appear as the right candidate in a physical situation where the probability distribution does not suit any of the previously defined forms.

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I. INTRODUCTION

Entropy is a measure of disorder or randomness, and assumes its maximal value when a system can be in a number of states randomly with equal probability, and is minimally zero when the system is in a given state, with no uncertainty in its description. Apart from this common feature shared by all definitions of entropy at two ends of the scale, variations are possible in particularizing the functional form in between [1]. They lead to different forms of the probability distributions for states with different energies or some other conserved attribute. Some turn up as extensive, where the entropy of a combination of systems is simply the sum of the entropies of the systems, as in the classical case of the Shannon form, while others can be defined to be not so. Renyi entropy [2] is different from Shannon, yet extensive, and hence the Shannon form is not unique with respect to the property of extensivity.

Tsallis entropy [3?] has attracted a lot of attention in recent years, not only on account of its conceptual and theoretical novelty, but also because it can be shown in specific physical cases [6, 7, 8] to be the relevant form where nonextensivity is expected on account of the interaction of the combined subsystems. In the proper limiting case it reduces to the standard Shannon entropy, indicating the consistency of the concept.

In this paper, however, we shall introduce entropy from a new perspective, which too will bear semblance to the normal form in the limit. We shall first present the rationale for this new definition and compare it briefly with the forms already being used. Then we shall find the form

of the probability distribution for this entropy, which we shall henceforth call s-entropy, as it will be seen to be related to the concept of rescaling of the phase space.

II. DEFINING THE NEW ENTROPY

Let us consider a register of only one letter. Let p_i be the set of probabilities for each of the N letters A_i that can occupy this position. We are here using the language of information theory, as used for example, in the Shannon Coding Theorem, though it is trivially extensible to states i of a single state of an ensemble where the individual systems can be in any N states with probabilities p_i .

Let us now consider a small deformation of the register to a new size so that it can accommodate $q = 1 + \Delta q$ letters. The probability that the whole new phase space is occupied by the letter p_i is now p_i^q by the corresponding AND operation and hence the probability that the new deformed cell is occupied by any of the pure letters A_i is

$$N(q) = \sum_i p_i^q \quad (1)$$

For $q > 1$ this would give a shortfall from the original total probability of unity for $q = 1$. It is obvious that the shortfall, which we denote by

$$M(q) = 1 - \sum_i p_i^q \quad (2)$$

represents the total probability that the mixed cell has a mixture of A_i and some other A_j fractionally, since the total probability that the cell is occupied by one or

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more (fractional included) letters must be unity. Hence the mixing probability $M(q)$ is actually a measure of the disorder introduced by increasing the cell scale from unity to $1 + \Delta q$.

The introduction of fractional values of cell numbers can be taken in the same spirit as defining the fractal (Hausdorff) dimensions of curves, and in complex systems there have been studies of diffusion [9] and percolation in complex systems with effectively fractional dimensions for fluids where the special geometric constraints translate into a change in the dimension of the corresponding space to an apparently nonintuitive fractional dimension. In coding theories for optimal transmission of information [10], we come across Huffman coding, where the optimum alphabet size may be formally a fraction, though for practical purposes it may be changed to the nearest higher integer. In probabilistic optimization, we may therefore consider a fractional size of the registrar, or equivalently, an integral number of cells in the registrar with fractional sized cells to accommodate a given amount of information. Probabilistic optimization in place of the deterministic parameterization of classical Shannon information theory [10] becomes inevitable in quantum computing contexts, and hence our use of the fractional cell sizes may be a classical precursor of the inevitable departure from stringent Shannon-type concepts.

For an alphabet of m letters we define the entropy from the information content of the registrar by

$$m^{S(q)\Delta q} = m^{(M(q+\Delta q)-M(q))} \quad (3)$$

so that the entropy indicates an effective change in the mixing probability due to an infinitesimal change in the cell-size of the registrar.

This leads to

$$S(q) = dM(q)/dq \quad (4)$$

In other words

$$S(q) = - \sum_i p_i^q \log p_i \quad (5)$$

We have some material at the other place. This differential form is analogous to but different from the Tsallis form

$$S_T(q) = - \sum_i (1 - p_i^q)/(1 - q) \quad (6)$$

where there is an apparent singularity at $q = 1$ which is the Shannon limit. The difference between the Tsallis expression and ours becomes clearer if we express entropy as the expectation value of the (generalized or ordinary) logarithm.

$$S_T(q) = \langle \text{Log}_q p \rangle \quad (7)$$

where the generalized q -logarithm is defined as

$$\text{Log}_q p_i = 1 - p_i^{(q-1)}/(1 - q) \quad (8)$$

The expectation value is defined in terms of the simple probability distribution

$$\langle O \rangle = \sum_i p_i O_i \quad (9)$$

In our case we define the expectation value with respect to the deformed probability corresponding to the extended cell, while keeping the usual logarithm

$$S_s(q) = \langle \log p \rangle_q \quad (10)$$

with

$$\langle O \rangle_q = \sum_i p_i^q O_i \quad (11)$$

In the limit $q \rightarrow 1$ Log_q approaches the normal logarithm, and hence Tsallis entropy coincides with Shannon entropy and also as $p_i^q \rightarrow p_i$ we too get the normal Shannon entropy.

The Renyi entropy is defined by

$$S_R(q) = \log(\sum_i p_i^q)/(1 - q) \quad (12)$$

Like Shannon entropy this one is also extensive, i.e. simply additive for two subsystems for any value of q . To get Shannon entropy uniquely one needs [11] a slightly different formulation of the extensivity axiom

$$S_{1+2} = S_1 + \sum_i p_{1i} S_2(i), \quad (13)$$

where $S_2(i)$ is the entropy of subsystem 2 given subsystem 1 is in state i .

III. PROBABILITY DISTRIBUTION FOR THE NEW ENTROPY

The p_i can be obtained in terms of the energy of the states, or possibly also other criteria in the usual way by maximizing the entropy with constraints

$$\sum_i p_i - 1 = 0 \quad (14)$$

and

$$\sum_i p_i E_i - U = 0 \quad (15)$$

The solution of the optimization equation gives for energy E_i the probability p_i

$$p_i = \left(\frac{-qW(z)}{(a + bE)(q-1)} \right)^{1/(1-q)} \quad (16)$$

where

$$z = -e^{(q-1)/q}(a + bE)(q-1)/q \quad (17)$$

and $W(z)$ is the Lambert function defined by

$$z = we^w \quad (18)$$

Here a and b are constants coming from the Lagrange's multipliers for the two constraints and are related to the overall normalization and to the relative scale of energy, i.e. to temperature ($1/(kT)$) as in the Shannon case where we get the Gibbs expression for p_i . In the Tsallis case p_i has the well-known value

$$p_i = (a + b(q-1)E)^{1/(1-q)} \quad (19)$$

which is easily seen to reduce to Shannon form for $q \rightarrow 1$.

After some algebra it can be shown that this form reduces to the Shannon form for $q \rightarrow 1$.

The nonextensivity of Tsallis entropy is seen easily by expanding

$$S_{1+2}^T = - \sum_{ij} p_i p_j (1 - p_i^q p_j^q) / (1 - q)^2 \quad (20)$$

$$= S_1^T + S_2^T + (1 - q) S_1^T S_2^T \quad (21)$$

For Renyi entropy we have the simple additive relation

$$S_{1+2}^R = S_1^R + S_2^R \quad (22)$$

In case of the new entropy

$$S_{1+2}^s = S_1^s + S_2^s + M_2(q) S_1^s + M_1(q) S_2^s \quad (23)$$

where the M_a are the mixing probability of states for subsystem a as defined in Eq.2.

IV. NUMERICAL COMPARISON

In Fig.1 we show the variation of the probability function for different E at different q values.

We note that the pdf drops increasingly rapidly for higher values of q , and is quite different in shape and in magnitude at high energy values from the Gibbs exponential distribution. A variation of even 10% from the

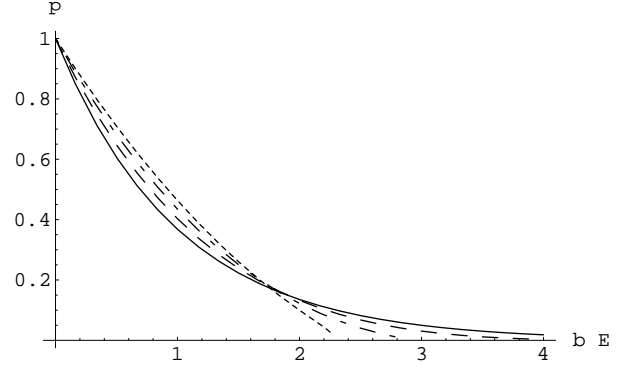


FIG. 1: Comparison of the pdf for the new entropy for values of $q = 1, 1.1, 1.2$ and 1.3 . The solid line is for $q = 1$, i.e. the Gibbs exponential distribution and the lines are in the order of q

standard value of $q = 1$ can cause a quite discernible change in the pdf and should be observable in experimental contexts fairly easily. At $q = 1.3$, the shape is almost linear.

In Fig.2 and Fig.3 we show the comparison of Tsallis pdf and the pdf for the new entropy for the same values of q , 1.1 in the former and 1.3 in the latter. We notice that for larger q values the new entropy gives much stiffer probability functions departing substantially from the Tsallis pdf's.

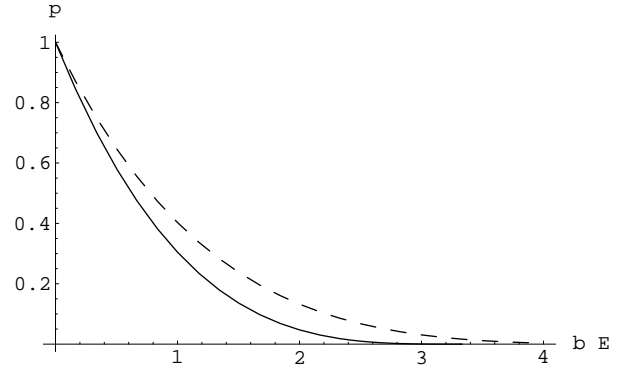


FIG. 2: Comparison of pdf's for Tsallis nonextensive entropy and the new entropy presented here, for $q = 1.1$

V. CONCLUSIONS

We see that the new entropy presented here based on the simple concept of the amount of mixing of states freedom introduced per unit cell of phase space leads to a nonextensive form different from any of the presently studied entropies. It leads to a complicated, but still integrable form of the pdf which departs substantially from Tsallis entropy. This entropy is also nonextensive in a fashion different from Tsallis entropy, though like

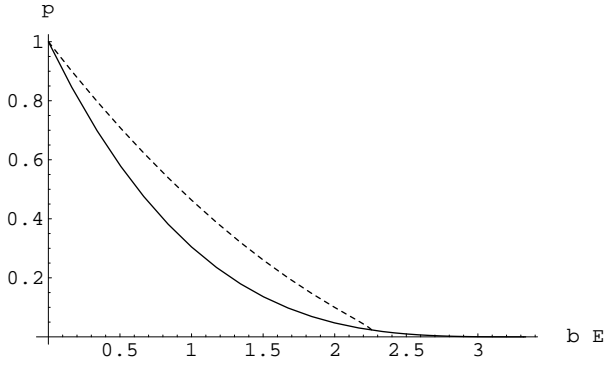


FIG. 3: The same as Fig. 2, but for a higher $q = 1.3$

Tsallis it too becomes extensive trivially in the limit $q \rightarrow 1$, as expected.

It would now be interesting to find a physical situation where such an entropy arises from first principles, though like some initial phenomenological studies of Tsallis entropy it can be also used as a parametrization scheme with q as a parameter to fit experimental data. The stiffness of any data may point to its preferability to Tsallis-type entropies.

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